Bittinger 12.6 & 12.7 : 3 lessons

12.6-1st Exponential Equations

Solve $b^x = a$ for x.

12.6-2nd Logarithmic Equations

Solve $\log_b x = a$ for x.

12.6-3rd Mixed practice

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Solve each equation.

a) Solve exactly.

b) Approximate solution to a) to 4 decimal places.

① $3^x = 7$

a) step 1: Take logs both sides using a log of base that you choose: base 10, base e, base 3.

BASE 10 Method	BASE e Method	Base 3 Method
$\log 3^x = \log 7$	$\ln 3^x = \ln 7$	$\log_3 3^x = \log_3 7$

step 2: Use log property to move exp x to front.

$\log_b x^k = k \log_b x$ (move exp) $\qquad \log_b b^x = x$ (inverse property)

$x \cdot \log 3 = \log 7$	$x \cdot \ln 3 = \ln 7$	$x \cdot \log_3 3 = \log_3 7$

step 3: Isolate x.

Notice $\log_3 3 = 1$

$\boxed{x = \dfrac{\log 7}{\log 3}}$	$\boxed{x = \dfrac{\ln 7}{\ln 3}}$	$\boxed{x = \log_3 7}$

b). With base 10 and base e, plug directly into GC.
With base 3, use change of base formula

$$\log_3 7 = \frac{\log 7}{\log 3} \quad \text{or} \quad \frac{\ln 7}{\ln 3}$$

look at 5 places

1.77124

$\approx \boxed{1.7712}$

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(2) $e^{6x} = 5$

a) $\boxed{\text{BASE 10 Method}}$

$\log e^{6x} = \log 5$

$6x \log e = \log 5$

$x = \dfrac{\log 5}{6 \log e}$

$\boxed{\text{Base e Method}}$ * preferred

$\ln e^{x} = x$
inverse property

$\ln e^{6x} = \ln 5$

$6x \cdot \ln e = \ln 5$

* Notice *
$\ln e = 1$

$6x = \ln 5$

$$\boxed{x = \frac{\ln 5}{6} = \frac{1}{6} \ln 5 = \ln \sqrt[6]{5}}$$

b) $x \approx .26823$

$\boxed{x \approx .2682}$

(3) $10^{2x+3} = 7$

$\boxed{\text{BASE 10 Method}}$ * preferred

$\log 10^{x} = x$
inverse property

a) $\log 10^{2x+3} = \log 7$

$(2x+3) \cdot \log 10 = \log 7$

* Notice $\log 10 = 1$ *

$2x+3 = \log 7$

$2x = -3 + \log 7$

$$\boxed{x = -\frac{3}{2} + \frac{\log 7}{2}}$$

$$\boxed{x = -\frac{3}{2} + \log \sqrt{7}}$$

$\boxed{\text{Base e Method}}$

$\ln 10^{2x+3} = \ln 7$

$(2x+3) \ln 10 = \ln 7$

$2x+3 = \dfrac{\ln 7}{\ln 10}$

$2x = -3 + \dfrac{\ln 7}{\ln 10}$

$$\boxed{x = -\frac{3}{2} + \frac{1}{2} \frac{\ln 7}{\ln 10}}$$

b) $x \approx -1.07745$

$\boxed{x = -1.0775}$

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Aside: Are these expressions equal?

a) $\frac{\ln 5}{6}$ b) $\ln\left(\frac{5}{6}\right)$ c) $\frac{\ln 5}{\ln 6}$

Try in GC:

a) $\frac{\ln 5}{6}$ = ln(5) / 6 $\approx .2682$

b) $\ln\left(\frac{5}{6}\right)$ = ln(5/6) $\approx -.1823$

c) $\frac{\ln 5}{\ln 6}$ = ln(5) / ln(6) $\approx .8982$

The short answer? No. They are totally different numerical results.

But what are these expressions if we use log properties or log facts?

b) $\ln\left(\frac{5}{6}\right) = \ln(5) - \ln(6)$ log property

c) $\frac{\ln 5}{\ln 6} = \log_6 5$ change of base

a) $\frac{\ln(5)}{6} = \frac{1}{6} \cdot \ln(5) = \ln(5^{1/6})$ log property.

This one is like $\frac{x}{6} = \frac{1}{6}x$

The moral: Be very careful where you close parentheses and where you write fraction bars.

Math 70

(4) $8 - 14 \cdot 5^{2x+1} = 1$

a) Step 0: Isolate the exponential 5^{2x+1}

$$-14 \cdot 5^{2x+1} = 1 - 8$$

$$\frac{-14 \cdot 5^{2x+1}}{-14} = \frac{-7}{-14}$$

$$5^{2x+1} = \frac{1}{2}$$

Step 1: Take logs, Step 2: Use log properties. Step 3 isolate x

BASE 10 Method

$$\log 5^{2x+1} = \log\left(\tfrac{1}{2}\right)$$

$$(2x+1) \cdot \log 5 = \log\left(\tfrac{1}{2}\right)$$

$$2x+1 = \frac{\log\left(\frac{1}{2}\right)}{\log(5)}$$

$$2x = -1 + \frac{\log\left(\frac{1}{2}\right)}{\log(5)}$$

$$\boxed{x = \frac{-1}{2} + \frac{1}{2}\frac{\log\left(\frac{1}{2}\right)}{\log(5)}}$$

Base e Method

$$\ln 5^{2x+1} = \ln\left(\tfrac{1}{2}\right)$$

$$(2x+1)\ln 5 = \ln\left(\tfrac{1}{2}\right)$$

$$2x+1 = \frac{\ln\left(\frac{1}{2}\right)}{\ln(5)}$$

$$2x = -1 + \frac{\ln\left(\frac{1}{2}\right)}{\ln(5)}$$

$$\boxed{x = \frac{-1}{2} + \frac{1}{2}\frac{\ln\left(\frac{1}{2}\right)}{\ln(5)}}$$

Base 5 Method

$$\log_5 5^{2x+1} = \log_5\left(\tfrac{1}{2}\right)$$

$$2x+1 = \log_5\left(\tfrac{1}{2}\right)$$

$$2x = -1 + \log_5\left(\tfrac{1}{2}\right)$$

$$\boxed{x = \frac{-1}{2} + \frac{1}{2}\log_5\left(\tfrac{1}{2}\right) \qquad x = \frac{-1}{2} + \log_5\sqrt{\tfrac{1}{2}}}$$

b) $-.71533$

$\boxed{-.7153}$

Math 70 Exponential Equations

1) How long does it take an investment for \$2000 to double if it is invested at 5% interest compounded quarterly? Round to the nearest tenth of a year.

$A=P\left(1+\frac{r}{n}\right)^{(nt)}$

$P=2000$
$A=4000$
$r=.05$
$n=4$
$t=?$

$4000=2000\left(1+\frac{.05}{4}\right)^{(4t)}$

solve for t ⇒ exponent

step 0: Isolate exponential

$\frac{4000}{2000}=\left(1+\frac{.05}{4}\right)^{4t}$

$2=(1.0125)^{4t}$

step 1: Take log of both sides.

$\log 2=\log(1.0125)^{4t}$

Step 2: log property

$\log 2 = 4t\cdot\log(1.0125)$

step 3: Isolate t.

$\frac{\log 2}{4\log(1.0125)}=t$

$t=13.94$

$\boxed{13.9\text{yr}}$

2) Suppose that you invest \$1500 at an annual rate of 8% compounded monthly.

a. Make a table giving the value of the investment at the end of each year for the next 14 years.

t	A
1	\$1624.50
2	\$1759.30
3	\$1905.40
4	\$2063.50
5	\$2234.80
6	\$2420.30
7	\$2621.10

t	A
8	\$2838.70
9	\$3074.30
10	\$3329.50
11	\$3605.80
12	\$3905.10
13	\$4229.20
14	\$4580.20

$A=P\left(1+\frac{r}{n}\right)^{nt}$

$A=1500\left(1+\frac{.08}{12}\right)^{12t}$

do not round! > frac

$A=1500\left(\frac{151}{150}\right)^{12t}$

b. How long does it take the investment to triple? (Exact answer.)

$1500\times 3=4500$ ⇒ from table, between 13 and 14 years

$A=4500$
$P=1500$
$r=.08$
$n=12$
$t=$ unknown

$4500=1500\left(1+\frac{.08}{12}\right)^{12t}$

$\frac{4500}{1500}=\left(\frac{151}{150}\right)^{12t}$

$3=\left(\frac{151}{150}\right)^{12t}$

$\log 3=\log\left(\frac{151}{150}\right)^{12t}$

$\log 3 = 12t\log\left(\frac{151}{150}\right)$

$\boxed{\frac{\log 3}{12\log\left(\frac{151}{150}\right)}=t}$ exact answer

$\boxed{t\approx 13.77\text{ yrs}}$

3) The formula $P = 14.7e^{-0.21x}$ gives the average atmospheric pressure P, in pounds per square inch, at an altitude x, in miles above sea level. Use this formula to find the elevation of a Delta jet if the atmospheric pressure outside the jet is 7.5 pounds per square inch. Round to the nearest tenth.

$7.5 = 14.7e^{-0.21x}$ find x

$\frac{7.5}{14.7} = e^{-0.21x}$

$\frac{25}{49} = e^{-0.21x}$

$\ln\left(\frac{25}{49}\right) = \ln\left(e^{-0.21x}\right)$

$\ln\left(\frac{25}{49}\right) = -0.21x$

$x = \frac{\ln\left(\frac{25}{49}\right)}{-0.21} \approx 3.2044$

$\boxed{\text{3.2 miles above sea level}}$

4) In 2010, the population of Illinois was approximately 12,830,000 and increasing by relative growth rate 0.5% annually. Assume that the population y continues to increase according to $y = y_0e^{kt}$, where k is the annual relative growth rate and y_0 is the initial population. Predict how many years after which the population will be 13,500,000.

initial population $12{,}830{,}000 = y_0$

percent increase = $0.5\% = 0.005 = k$

model $y = 12830000\,e^{.005t}$

"predict how many years" = find t.

population $13\,500\,000 = y$

$13\,500\,000 = 12\,830\,000\,e^{0.005t}$ ← This is an exponential equation.
- isolate the exponential
- take an appropriate log.

$\frac{13\,500\,000}{12\,830\,000} = e^{0.005t}$

$\frac{1350}{1283} = e^{0.005t}$ ← base e ⇒ use natural log, ln

$\ln\left(\frac{1350}{1283}\right) = \ln e^{0.005t} = 0.005t$ ← $\log_b b^x = x$ means $\ln e^x = x$

isolate t by ÷ 0.005

$\boxed{\frac{\ln\left(\frac{1350}{1283}\right)}{0.005} = t}$ exact answer

GC: $t \approx 10.1807$ during 2020 it passes 13.5 million

$\boxed{2021}$ entire year